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Dynamo action associated with random waves in a rotating stratified fluid

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A random superposition of waves in a rotating, stratified, electrically conducting fluid leads to dynamo action in the sense that it yields a mean electric field having a component parallel to the mean magnetic field ('α-effect'). Using Fourier analysis methods, we derive an explicit expression for the mean electric field. The α-effect has tensor form. We obtain a finite α-tensor even in a case of vanishing mean helicity. The result is discussed in the context of the solar turbulent dynamo.

1. Introduction

The subject of mean field electrodynamics was initiated by Steenbeck, Krause & Rädler (1966). For an electrically conducting fluid, which is in random motion, $\mathbf{u}$, and is permeated by an induced fluctuating magnetic field, $\mathbf{h}$, they found that the mean electric field, $\mathscr{E} = \overline{\mathbf{u} \times \mathbf{h}}$, has a component parallel or anti-parallel to the mean magnetic field, $\bar{\mathbf{h}}$, provided the motion is sufficiently anisotropic. It is well known that such an effect ('α-effect') causes a dynamo instability: an initially small mean field will grow exponentially in time (Parker 1955).

Moffatt (1970*a*, *b*, 1972, 1974, 1978) applied Fourier analysis methods to the problem and thus was able to treat cases where $\mathbf{u}$ consists of a superposition of random waves. Here the time dependence of each wave is given by a dispersion relation and the space and time correlations of the velocity components are therefore no longer separable. However, in order to obtain a non-vanishing mean electric field, Moffatt (1970*b*, 1972) made the assumption that there is a preferred direction of wave propagation. In the present paper we shall drop this assumption and instead investigate the dynamo action of waves in a rotating *stratified* medium. We thus have, in addition to rotation, a natural means to introduce anisotropy into the random velocity field. This approach will also remove the particular behaviour of Moffatt's result in the limit of slow rotation, where he obtained a finite α-effect, while a result proportional to the rate of rotation would seem more plausible and would also be in agreement with the earlier results of Steenbeck *et al.* (1966) and Krause (1968) who also investigated this limit.

In § 2 we shall consider the linearized equations governing the propagation of waves in an isothermal atmosphere under the influence of rotation. Since rotation will be considered only as a weak influence, the emphasis will be on acoustic and internal gravity waves rather than on inertial waves as in Moffatt's work. An expression for the spectrum tensor of these waves will be derived in § 3, and in § 4 we shall discuss the helicity, $\mathbf{u}.\operatorname{curl}\mathbf{u}$ of our random superposition of wave modes. We shall see that a velocity field with zero mean helicity can still provide a mean electric field leading to dynamo action. After a discussion of the induced magnetic field in § 5 we derive, in § 6, the tensor α_{ij} which relates $\mathscr{E}$ to the mean field. Results for the limiting cases of very small and very large wavelength (compared to the scale height of density stratification) will be communicated in § 7, both in the limits of small and large electrical conductivity. Finally, in § 8, we discuss an illustrative two-dimensional example of a large-scale (mean) magnetic field excited by the dynamo mechanism studied in the preceding sections.

2. Waves in a rotating stratified atmosphere

We divide the Eulerian density and pressure variables into their average and fluctuating parts, viz.

$$\rho_0 = \bar{\rho}+\rho, \tag{1}$$

$$p_0 = \bar{p}+p. \tag{2}$$

We adopt an equilibrium state with no motion, i.e. $\bar{\mathbf{u}} = 0$. The linearized equations of momentum, mass and energy conservation, in a rotating frame of reference, are then

$$\bar{\rho}\frac{\partial \mathbf{u}}{\partial t} = -\nabla p+\rho\mathbf{g}-2\bar{\rho}(\mathbf{\Omega}'\times\mathbf{u}), \tag{3}$$

$$\frac{\partial \rho}{\partial t} = -\nabla.(\bar{\rho}\mathbf{u}), \tag{4}$$

$$\frac{\partial p}{\partial t}+\mathbf{u}.\nabla\bar{p} = c^2\left(\frac{\partial \rho}{\partial t}+\mathbf{u}.\nabla\bar{\rho}\right). \tag{5}$$

We consider a plane parallel geometry (approximately valid for a thin spherical shell), where $\mathbf{g} = -g\hat{\mathbf{z}}$ is the acceleration of gravity; $\mathbf{\Omega}'$ is the vector of rotation and $c = (\bar{p}\gamma/\bar{\rho})^{\frac{1}{2}}$ is the Laplacian velocity of sound. In (3) we have neglected the centrifugal force, which is reasonable if the rotation is slow in the sense $R\Omega'^2/g \ll 1$, which is satisfied for most stars. The Coriolis force is however retained; a consistency condition is therefore $\rho/\bar{\rho} \ll 2u/R\Omega'$. Since we consider the growth of infinitesimal magnetic fields we have neglected the Lorentz force, i.e. our study is kinematic. Equation (5) describes the adiabatic case; the ratio of specific heats, γ, will be considered as a constant. We shall also assume that the unperturbed stratification is isothermal, so that

$$\bar{\rho}, \bar{p} \propto \exp(-z/H), \tag{6}$$

where z is the vertical co-ordinate and the scale height, H, is a constant:

$$H = c^2/\gamma g. \tag{7}$$

We now introduce the 'field variables' (Eckart 1960, p. 55)

$$\mathbf{U} = \mathbf{u}(\bar{\rho}c)^{\frac{1}{2}}, \quad P = p(\bar{\rho}c)^{-\frac{1}{2}}, \quad S = (p - c^2\rho)(\gamma-1)^{-\frac{1}{2}}(\bar{\rho}c)^{-\frac{1}{2}}. \tag{8}$$

The variable S differs from Eckart's Q by a factor N', the Brunt–Väisälä frequency:

$$N' = (\gamma-1)^{\frac{1}{2}} g/c. \tag{9}$$

Equations (3)–(5) are thus reduced to a system with constant coefficients, and the solutions have exponential form:

$$(\mathbf{U}, P, S) = (\tilde{\mathbf{U}}, \tilde{P}, \tilde{S}) \exp[i(\mathbf{k}'.\mathbf{x} - \omega' t)]. \tag{10}$$

We use normalized frequencies, ω and N, and wave and rotation vectors, $\mathbf{k}$ and $\mathbf{\Omega}$, defined by

$$\omega' = c\omega/2H, \quad N' = cN/2H, \quad \mathbf{k}' = \mathbf{k}/2H, \quad \mathbf{\Omega}' = c\mathbf{\Omega}/4H, \tag{11}$$

and obtain the algebraic system

$$\omega\tilde{\mathbf{U}} = \mathbf{k}\tilde{P} - \tfrac{1}{2}i\Gamma\tilde{P}\hat{\mathbf{z}} + iN\tilde{S}\hat{\mathbf{z}} - i\mathbf{\Omega}\times\tilde{\mathbf{U}}, \tag{12}$$

$$\omega\tilde{S} = -iN\tilde{\mathbf{U}}.\hat{\mathbf{z}}, \tag{13}$$

$$\omega\tilde{P} = \tilde{\mathbf{U}}.\mathbf{k} + \tfrac{1}{2}i\Gamma\mathbf{U}.\hat{\mathbf{z}}, \tag{14}$$

where $\Gamma = 2(2-\gamma)/\gamma$. This system is self-adjoint (because of the neglect of all dissipative terms in our original equations), and therefore all frequencies ω are real. One of them is zero and belongs to a steady solution which is horizontally in geostrophic, and vertically in hydrostatic, equilibrium; here we shall not consider this solution further. The four remaining frequencies are obtained from the following dispersion relation:

$$D = \omega^4 - (1 + k^2 + \Omega^2)\omega^2 + \Gamma\mathbf{k}.(\mathbf{\Omega}\times\hat{\mathbf{z}})\omega + (\mathbf{k}.\mathbf{\Omega})^2 + (\mathbf{\Omega}.\hat{\mathbf{z}})^2 + N^2(\mathbf{k}\times\hat{\mathbf{z}})^2 = 0. \tag{15}$$

Equation (15) is the condition that the system (12)–(14) has non-trivial solutions. Since

$$(\mathbf{k}.\mathbf{\Omega})^2 + (\mathbf{\Omega}.\hat{\mathbf{z}})^2 + N^2(\mathbf{k}\times\hat{\mathbf{z}})^2 \geqslant 0,$$

each of the four frequency branches has a well-defined sign. In the case of no rotation, we have the two acoustic wave branches

$$\omega_{0K\pm} = \pm\{\tfrac{1}{2}(1+k^2) + [\tfrac{1}{4}(1+k^2)^2 - N^2(\mathbf{k}\times\hat{\mathbf{z}})^2]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{16}$$

and the two internal gravity wave branches

$$\omega_{0G\pm} = \pm\{\tfrac{1}{2}(1+k^2) - [\tfrac{1}{4}(1+k^2)^2 - N^2(\mathbf{k}\times\hat{\mathbf{z}})^2]^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{17}$$

For part of the subsequent discussion it is convenient to introduce polar co-ordinates θ, ϕ in the $\mathbf{k}$ space, with $\hat{\mathbf{z}}$ as axis; if θ_0 is the angle between $\hat{\mathbf{z}}$ and the axis of rotation, we define the x and y directions such that

$$\mathbf{\Omega} = \Omega(0, -\sin\theta_0, \cos\theta_0) \tag{18}$$

and

$$\mathbf{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{19}$$

(We may imagine that the system of rectangular co-ordinates used previously sits on a sphere, with the origin at co-latitude θ_0, and the x and y directions pointing

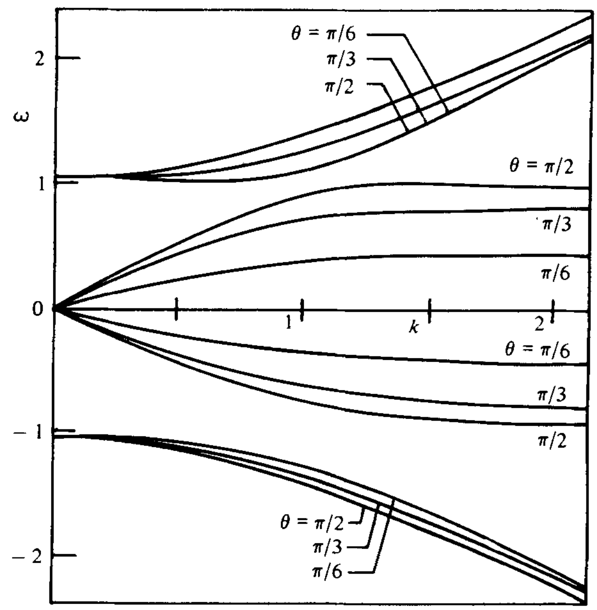

FIGURE 1. Dispersion relation $\omega(k)$ for $\Omega = \frac{1}{3}$ and a horizontal axis of rotation; θ is the inclination of the wave vector from the vertical. Notice the asymmetry between positive and negative frequencies, which does not occur for $\Omega = 0$. Parameters are $\gamma = \frac{5}{3}$, so that $N^2 = 0{\cdot}96$, and $\theta_0 = \frac{1}{2}\pi$, $\phi = 0$.

towards west and south respectively. Thus θ is the angle between **k** and the local vertical, and ϕ is the angle between west and the projection of **k** onto the local horizontal plane.)

The symmetry of the two acoustic and internal gravity wave frequencies with respect to $\omega = 0$ is destroyed when $\Omega \neq 0$. This is illustrated in figure 1 which shows the four branches of $\omega(\mathbf{k})$, evaluated numerically for $\theta_0 = \frac{1}{2}\pi$, $\Omega = \frac{1}{3}$, $\phi = 0$, and various values of θ; the values of θ_0 and ϕ were chosen such that the asymmetry introduced by rotation is most clearly exhibited. The rotational effect is also illustrated in figures 2 and 3 where the surfaces ω = constant in the **k** space are shown. These surfaces are ellipsoids for the acoustic modes, and hyperboloids of one sheet for the internal gravity modes. Their equations are obtained from the dispersion relationship

$$k_x^2(N^2-\omega^2)+k_y^2(N^2+\Omega_y^2-\omega^2)+k_z^2(\Omega_z^2-\omega^2)+2k_y k_z\Omega_y\Omega_z + \Gamma k_x\Omega_y\omega+\omega^4-(1+\Omega^2)\,\omega^2+\Omega_z^2 = 0. \quad (20)$$

The centre of these surfaces is shifted along the k_x axis to

$$k_{xc} = -\tfrac{1}{2}\Gamma\Omega_y\omega/(N^2-\omega^2), \quad (21)$$

and the whole surfaces are rotated about the k_x axis by an angle ϵ, where

$$\tan 2\epsilon = \frac{2\Omega_y\Omega_z}{N^2+\Omega_y^2-\Omega_z^2}. \quad (22)$$

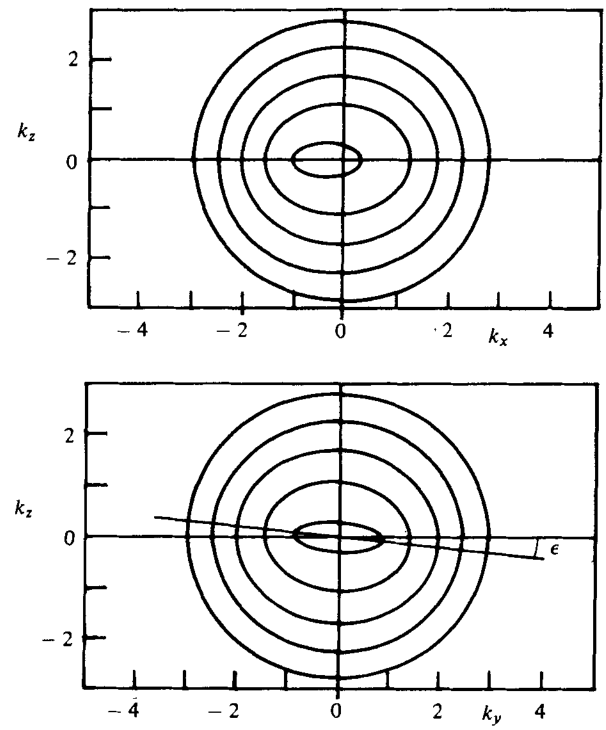

FIGURE 2. Surfaces ω = constant for acoustic modes, projected to the plane $k_y = 0$ (top) and to the plane $k_x = 0$ (bottom). Parameters are $\gamma = \frac{5}{3}$, and $\Omega_z = -\Omega_y = \frac{1}{2}$, so that $\epsilon \approx -6{\cdot}5°$; the frequencies are $\omega = 1{\cdot}1$, $1{\cdot}5$, $2{\cdot}0$, $2{\cdot}5$ and $3{\cdot}0$, increasing outwards.

Inspection of the dispersion relation (15) shows that the term responsible for the asymmetry between the positive and negative branches of ω is proportional to $\mathbf{k}.(\mathbf{\Omega}\times\hat{\mathbf{z}})$. This asymmetry therefore vanishes for waves propagating in a meridional direction, where $\mathbf{k}$, $\mathbf{\Omega}$ and $\hat{\mathbf{z}}$ are co-planar. We shall give the rotational modification of the frequencies (16) and (17) explicitly in § 7 for a number of limiting cases.

3. The spectrum tensor

The spectrum tensor associated with the random field **U** is defined by

$$\Phi_{ij}(\mathbf{k})\,\delta(\mathbf{k}-\mathbf{l}) = \overline{\tilde{U}_i^*(\mathbf{k})\,\tilde{U}_j(\mathbf{l})}, \qquad (23)$$

where $\tilde{\mathbf{U}}$ is given by (10). Since the frequencies ω of our waves are determined by the dispersion relation as functions of $\mathbf{k}$, we do not consider $\mathbf{k}$ and ω as independent variables. We may consider the overbar as indicating an average over time, or an ensemble average.

We intend to determine Φ_{ij} from our basic equations. To this end, we eliminate $\tilde{P}$ and $\tilde{S}$ from (12)–(14) and obtain, in matrix notation,

$$M_{ij}\tilde{U}_j = 0, \qquad (24)$$

where

$$M_{ij} = k_ik_j+\hat{z}_i\hat{z}_j-\omega^2\delta_{ij}+i\epsilon_{ijs}[\tfrac{1}{2}\Gamma(\mathbf{k}\times \quad +\omega\Omega_s]. \qquad (25)$$

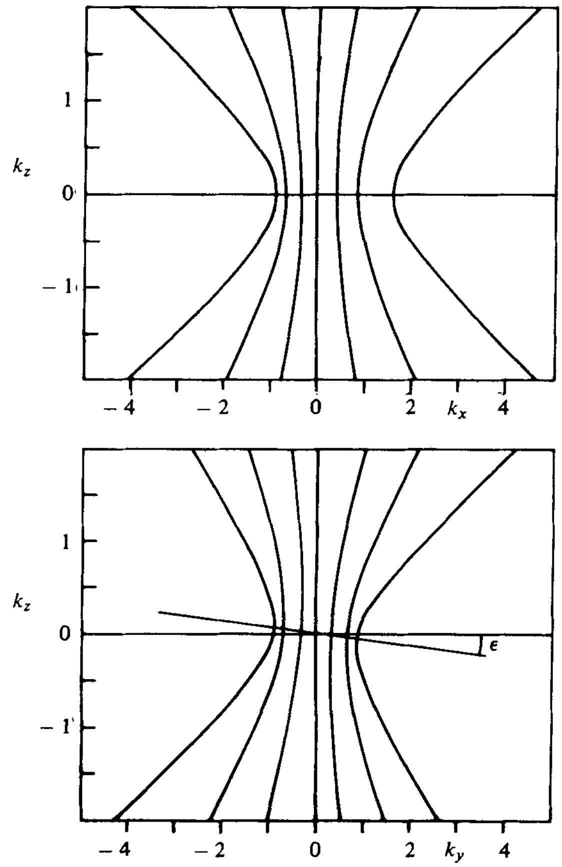

FIGURE 3. Surfaces ω = constant for internal gravity modes. The frequencies are $\omega = 0{\cdot}45$, $0{\cdot}7$ and $0{\cdot}9$, increasing with distance from the k_z axis; otherwise as figure 2.

This matrix is Hermitian; we therefore have

$$M^*_{ij}\tilde{U}^*_j = M_{ji}\tilde{U}^*_j = 0. \tag{26}$$

Now the matrix A_{ij} consisting of the co-factors of M_{ij} has precisely the properties required for our spectrum tensor: for any fixed i, A_{ij} is a solution to (24), for any fixed j it is a solution to (26), and A_{ij} is Hermitian. Therefore the spectrum tensor must be proportional to A_{ij}; the coefficient, S, may however depend on $\mathbf{k}$ (or on k, θ and ϕ):

$$\Phi_{ij} = S(k,\theta,\phi)\,A_{ij}. \tag{27}$$

From (25) we obtain A_{ij}, so that

$$\begin{aligned}\Phi_{ij} = S\{&\omega^2(\omega^2-1-k^2)\,\delta_{ij}+\omega^2(\hat{z}_i\hat{z}_j+k_ik_j-\Omega_i\Omega_j)+N^2(\mathbf{k}\times\hat{\mathbf{z}})_i(\mathbf{k}\times\hat{\mathbf{z}})_j\\ &-\tfrac{1}{2}\Gamma\omega[\Omega_i(\mathbf{k}\times\hat{\mathbf{z}})_j+\Omega_j(\mathbf{k}\times\hat{\mathbf{z}})_i]-i\omega\epsilon_{ijs}[\tfrac{1}{2}\Gamma\omega(\mathbf{k}\times\hat{\mathbf{z}})_s+\omega^2\Omega_s\\ &-(\mathbf{\Omega}.\hat{\mathbf{z}})\,\hat{z}_s-(\mathbf{\Omega}.\mathbf{k})\,k_s]\}.\end{aligned} \tag{28}$$

The energy density spectrum $E(k')$ is defined by

$$E(k') = \frac{1}{2}\int_{-\pi}^{\pi}\int_0^{\pi}\Phi_{ii}k'^2\sin\theta\,d\theta\,d\phi \tag{29}$$

[Φ_{ij} and E have dimensions; the dimensional wavenumber $\mathbf{k}'$ must therefore be used in (29)]. We may express the undetermined function $S(\mathbf{k})$ in terms of the energy density spectrum if we assume that the trace Φ_{ii} does not depend on θ and ϕ, i.e. that kinetic energy is generated isotropically.

Then we have

$$E(k') = 2\pi k'^2 \Phi_{ii}, \tag{30}$$

and, by (28),

$$S = E(k')\,(2\pi k'^2)^{-1}\,[3\omega^4 - \omega^2(\Omega^2 + 2 + 2k^2) + N^2(\mathbf{k}\times\hat{\mathbf{z}})^2 + \Gamma\omega\mathbf{k}\,.\,(\mathbf{\Omega}\times\hat{\mathbf{z}})]^{-1}. \tag{31}$$

4. The helicity

Steenbeck *et al.* (1966) and Krause (1968) found an α-effect, i.e. a non-vanishing component of $\mathbf{u}\times\mathbf{h}$ in the direction of $\mathbf{h}$, for velocity fields $\mathbf{u}$ lacking reflection symmetry. For such fields the helicity

$$\mathscr{H} = \overline{\mathbf{u}\,.\,\mathrm{curl}\,\mathbf{u}} \tag{32}$$

does not vanish. We now evaluate the helicity for a random superposition of waves in a rotating stratified fluid. Using (8) and (10) our velocity field is

$$\mathbf{u} = (\bar{\rho}c)^{-\frac{1}{2}}\,\mathrm{Re}\int \tilde{\mathbf{U}}(\mathbf{k}')\exp i(\mathbf{k}'.\mathbf{x} - \omega' t)\,d\mathbf{k}', \tag{33}$$

where the integral is over the entire $\mathbf{k}'$ space and, in addition, for each $\mathbf{k}'$, the contributions from the four branches of $\omega(\mathbf{k}')$ must be added together. As in the case of no stratification, the helicity can be computed according to

$$\mathscr{H} = \frac{i\epsilon_{ikl}}{\bar{\rho}c(2H)^4}\int k_k\,\Phi_{il}\,d\mathbf{k} \tag{34}$$

(cf. Moffatt 1978, p. 160), since the term proportional to $\nabla\bar{\rho}$ does not contribute to the integral. Consistently with its definition (32) $\mathscr{H}$ is real as there are contributions only from the anti-symmetric part of the spectrum tensor. Inserting (28) we obtain

$$\mathscr{H} = \frac{1}{\bar{\rho}c(2H)^4}\int_0^\infty\int_{-\pi}^{\pi}\int_0^{\pi} S(\mathbf{k})\,[(\mathbf{\Omega}.\hat{\mathbf{z}})\,(\hat{\mathbf{z}}.\mathbf{k}) + (\mathbf{\Omega}.\mathbf{k})\,(k^2-\omega^2)]\,\omega k^2\sin\theta\,d\theta\,d\phi\,dk. \tag{35}$$

With the special form of the function S given by (31), and with the help of (18) and (19) we can write

$$\mathscr{H} = \frac{\Omega}{2\pi\bar{\rho}c(2H)^4}$$
$$\times\int_0^\infty\int_{-\pi}^{\pi}\int_0^{\pi}\frac{E(k)\,[(1-\omega^2+k^2)\cos\theta_0\cos\theta + (\omega^2-k^2)\sin\theta_0\sin\theta\sin\phi]\,k\omega\sin\theta\,d\theta\,d\phi\,dk}{2\omega^4 - (1+k^2)\,\omega^2 - \Omega^2\cos^2\theta_0 - k^2\Omega^2(\cos\theta_0\cos\theta - \sin\theta_0\sin\theta\sin\phi)^2}. \tag{36}$$

Here we once more made use of the dispersion relation in order to simplify the denominator. Now we readily see that $\mathscr{H}$ must be zero as the integrand of (36) changes its sign when we go from the direction (θ, ϕ) to the direction $(\pi-\theta, -\phi)$; the dispersion relation, and therefore ω, is invariant under this transformation. Since the intervals of integration are symmetric with respect to $\theta = \frac{1}{2}\pi$ and $\phi = 0$, the contributions from the two hemispheres in $\mathbf{k}$ space cancel each other. This is separately true for all four frequency branches.

Of course, the result of zero mean helicity depends on the assumption of isotropic energy generation made in the preceding section. If the two waves propagating in the (θ, ϕ) and in the $(\pi-\theta, -\phi)$ directions have equal amplitudes, their helicities, which have opposite sign, cancel.

5. The induced field fluctuations

We use the induction equation for the magnetic field fluctuation, $\mathbf{h}$, in its 'first-order smoothed' version, and with zero mean motion, $\bar{\mathbf{u}} = 0$. Thus

$$\partial \mathbf{h}/\partial t = \operatorname{curl}(\mathbf{u} \times \bar{\mathbf{h}}) + \lambda \Delta \mathbf{h}, \tag{37}$$

where λ is the electromagnetic diffusivity. The first-order smoothing approximation, where products of fluctuating velocities and fields are neglected, is valid if the velocity amplitude u is small either compared to the diffusive velocity $k\lambda$ or compared to the phase velocity ω/k of the wave (e.g. Moffatt 1978, p. 156). The former possibility is the small magnetic Reynolds number limit, which is not applicable to most astrophysical situations. The latter case is independent of the magnetic Reynolds number, but it must be kept in mind that the waves must have small amplitudes; in particular in the limit $\omega \to 0$ this poses a problem.

We write the velocity field, $\mathbf{u}$, in the form

$$\mathbf{u} = \operatorname{Re}(\tilde{\mathbf{u}} \exp i(\mathbf{K}.\mathbf{x} - \omega' t)), \tag{38}$$

where $\mathbf{K}$ is now a complex wave vector,

$$\mathbf{K} = \mathbf{k}' - \tfrac{1}{2} i \hat{\mathbf{z}}/H. \tag{39}$$

The solution to (37) is then

$$\mathbf{h} = \operatorname{Re}(\tilde{\mathbf{h}} \exp(i(\mathbf{K}.\mathbf{x} - \omega' t)) \tag{40}$$

and the complex amplitude of $\mathbf{h}$ is

$$\tilde{\mathbf{h}} = -\frac{\tilde{\mathbf{u}}(\mathbf{K}.\bar{\mathbf{h}}) - \bar{\mathbf{h}}(\mathbf{K}.\tilde{\mathbf{u}})}{\omega' + i\lambda K^2}, \tag{41}$$

where

$$K^2 = k'^2 - \tfrac{1}{4} H^{-2} - i\mathbf{k}'.\hat{\mathbf{z}}/H. \tag{42}$$

We see from (41) and (42) that the phase relationship between $\mathbf{h}$ and $\mathbf{u}$, which is crucial in determining the mean $\overline{\mathbf{u} \times \mathbf{h}}$, is influenced by the effects of finite electrical conductivity and of stratification. We shall see presently that both effects together leave us with a finite α-effect in spite of the vanishing mean helicity.

6. The α-tensor

For a superposition of waves as described by (33) and the associated superposition of induced field fluctuations (40) we calculate the mean electric field $\mathscr{E}$ according to

$$\mathscr{E} \equiv \overline{\mathbf{u} \times \mathbf{h}} = \tfrac{1}{2} \operatorname{Re} \int \overline{\tilde{\mathbf{u}} \times \tilde{\mathbf{h}}^*} \, d\mathbf{k}'. \tag{43}$$

Using the field variable **U** and relation (41), we may express this in terms of the spectrum tensor:

$$\mathscr{E}_i = \frac{\epsilon_{ijk}}{2\bar{\rho}c}\,\mathrm{Re}\int \frac{(\omega' + i\lambda K^2)\,K_l^*(\Phi_{lj}\bar{h}_k - \Phi_{kj}\bar{h}_l)}{|\omega' + i\lambda K^2|^2}\,d\mathbf{k}'. \tag{44}$$

We introduce the dimensionless number

$$Q = \lambda/(2Hc), \tag{45}$$

which is the ratio of the time required by a sound wave to propagate over the distance $2H$ to the diffusion time for a magnetic fluctuation of wavenumber $(2H)^{-1}$. Then we use again the dimensionless frequencies and wavenumbers defined by (11) and obtain

$$\mathscr{E}_i = \frac{\epsilon_{ijk}}{2\bar{\rho}c^2(2H)^3}\int \frac{\mathrm{Re}\,[(\Phi_{lj}\bar{h}_k - \Phi_{kj}\bar{h}_l)f_l]}{[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]^2 + Q^2(k^2-1)^2}\,d\mathbf{k}, \tag{46}$$

where

$$f_l = [\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]\,k_l - Q(k^2-1)\,\hat{z}_l + i\{Q(k^2-1)\,k_l + [\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]\,\hat{z}_l\}. \tag{47}$$

We now use expressions (28) and (31) for the spectrum tensor and write the mean electric field in the form

$$\mathscr{E}_i = \frac{\bar{h}_j}{8\pi\bar{\rho}c^2H} \times \int_0^\infty\int_{-\pi}^{\pi}\int_0^{\pi} \frac{E(k)\,(a_{ij} + c_{ij})\sin\theta\,d\theta\,d\phi\,dk}{[2\omega^4 - (1+k^2)\,\omega^2 - (\mathbf{k}\,.\,\boldsymbol{\Omega})^2 - (\boldsymbol{\Omega}\,.\,\hat{\mathbf{z}})^2]\,\{[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]^2 + Q^2(k^2-1)^2\}}. \tag{48}$$

The first factor of the denominator has been treated as in the corresponding formula for the helicity (36). The tensor in the numerator has been divided into its symmetric and anti-symmetric parts, a_{ij} and c_{ij}, respectively; straightforward algebra leads to

$$\begin{aligned} a_{ij} = {} & -2k_ik_j\,\omega Q(k^2-1)\,(\boldsymbol{\Omega}\,.\,\mathbf{k}) + [(\mathbf{k}\times\hat{\mathbf{z}})_i\,k_j + (\mathbf{k}\times\hat{\mathbf{z}})_j\,k_i]\,\tfrac{1}{2}\Gamma\omega^2 Q(k^2-1) \\ & - (k_i\hat{z}_j + k_j\hat{z}_i)\,\omega[Q(k^2-1)\,(\boldsymbol{\Omega}\,.\,\mathbf{z}) + [\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]\,(\boldsymbol{\Omega}\,.\,\mathbf{k})\} \\ & + [(\mathbf{k}\times\hat{\mathbf{z}})_i\,\hat{z}_j + (\mathbf{k}\times\hat{\mathbf{z}})_j\,\hat{z}_i]\,\tfrac{1}{2}\Gamma\omega^2[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})] \\ & - 2\hat{z}_i\hat{z}_j\,\omega[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]\,(\boldsymbol{\Omega}\,.\,\hat{\mathbf{z}}) + (\Omega_ik_j + \Omega_jk_i)\,\omega^3Q(k^2-1) \\ & + (\Omega_i\hat{z}_j + \Omega_j\hat{z}_i)\,\omega^3[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})], \end{aligned} \tag{49}$$

and

$$\begin{aligned} c_{ij} = {} & \epsilon_{ijl}[\![\omega[(\mathbf{k}\times\hat{\mathbf{z}})_l\,\tfrac{1}{2}\Gamma + \omega\Omega_l]\,\{(\boldsymbol{\Omega}\,.\,\mathbf{k})\,[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})] - Q(k^2-1)\,(\boldsymbol{\Omega}\,.\,\hat{\mathbf{z}})\} \\ & + k_l\,\omega^2\{Q(k^2-1)\,(\mathbf{k}\,.\,\hat{\mathbf{z}}) - [\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]\,(\omega^2-1)\} \\ & + \hat{z}_l\,\omega^2\{Q(k^2-1)\,(\omega^2-k^2) - (\mathbf{k}\,.\,\hat{\mathbf{z}})\,[\omega + 2Q(\mathbf{k}\,.\,\hat{\mathbf{z}})]\}]\!]. \end{aligned} \tag{50}$$

The anti-symmetric part of the tensor connecting $\mathscr{E}$ and $\tilde{\mathbf{h}}$ (the 'α-tensor') can be written in the form

$$\epsilon_{ijl}\,C_l, \tag{51}$$

and C_l is readily obtained from (48) and (50). This part is of little help in solving the dynamo problem: in the induction equation for the mean field, the vector **C** enters in the same form as the mean velocity, so **C** and (or) the mean field itself must be sufficiently asymmetric in order to overcome the classical antidynamo theorems, e.g. that of Cowling which excludes axi-symmetric or two-dimensional fields. Of course, the term would modify solutions of the mean field equation obtained with the help of the symmetric part, which we shall consider now.

First, we notice that in the case of no rotation the symmetric part vanishes, because the ϕ integral of (48) gives zero. If Q is large, i.e. the conductivity small, α_{ij} vanishes at least as Q^{-1}; some of its components vary as Q^{-2} for reasons of symmetry in $\mathbf{k}$ space, as we shall see presently.

The integral (48) depends on the directions $\mathbf{\Omega}$ and $\hat{\mathbf{z}}$. A general expression for the α-tensor is therefore

$$\begin{aligned}\alpha_{ij} = \alpha_1 \delta_{ij} + \alpha_2 \hat{z}_i \hat{z}_j + \alpha_3 \Omega_i \Omega_j + \alpha_4 (\Omega_i \hat{z}_j + \Omega_j \hat{z}_i) \\ + \alpha_5[(\mathbf{\Omega} \times \hat{\mathbf{z}})_i \hat{z}_j + (\mathbf{\Omega} \times \hat{\mathbf{z}})_j \hat{z}_i] + \alpha_6[(\mathbf{\Omega} \times \hat{\mathbf{z}})_i \Omega_j + (\mathbf{\Omega} \times \hat{\mathbf{z}})_j \Omega_i] \\ + \epsilon_{ijs}[\alpha_7 \hat{z}_s + \alpha_8 \Omega_s + \alpha_9 (\mathbf{\Omega} \times \hat{\mathbf{z}})_s]. \end{aligned} \tag{52}$$

The first term on the right-hand side constitutes a mean electric field in the direction of $\bar{\mathbf{h}}$. We may therefore identify α_1 with the α of Steenbeck *et al.* (1966). Since both $\hat{\mathbf{z}}$ and $\mathbf{\Omega}$ have no x component, we have $\alpha_1 = \alpha_{11}$, i.e., by (48) and (49),

$$\alpha_1 = \frac{Q}{8\pi\bar{\rho}c^2 H} \times \int_0^\infty \int_{-\pi}^{\pi} \int_0^{\pi} \frac{E(k)\,(k^2-1)\,\omega k^2[\Gamma\omega \sin\phi \cos\phi - 2(\mathbf{\Omega}.\mathbf{k})\cos^2\phi]\sin^3\theta\, d\theta\, d\phi\, dk}{[2\omega^4 - (1+k^2)\,\omega^2 - (\mathbf{k}.\mathbf{\Omega})^2 - (\mathbf{\Omega}.\hat{\mathbf{z}})^2]\{[\omega + 2Q(\mathbf{k}.\hat{\mathbf{z}})]^2 + Q^2(k^2-1)^2\}}. \tag{53}$$

We cannot apply the same argument here which led us to conclude that the helicity is zero. For, if θ and ϕ are replaced by $(\pi - \theta)$ and $(-\phi)$, the term

$$\omega + 2Q\mathbf{k}.\hat{\mathbf{z}} = \omega + 2Qk\cos\theta$$

in the denominator of (53) changes to $\omega - 2Qk\cos\theta$, i.e. the integrand is neither symmetric nor anti-symmetric under this transformation. The α-effect is therefore generally non-zero. It does however disappear in the limit of infinite conductivity, not only because of the factor Q in front of the integral (53), but also because the integrand turns anti-symmetric in this limit. For small Q, an expansion then shows that $\alpha_1 \propto Q^2$; as already mentioned we have $\alpha_1 \propto Q^{-2}$ when $Q \gg 1$, for an analogous reason. A number of results of such expansions will be presented in the following section.

We have chosen to identify α_1 with Steenbeck and Krause's α because it represents an electric field parallel to $\bar{\mathbf{h}}$. We could also set $\alpha = \frac{1}{3}\alpha_{ii}$, equally consistent with Steenbeck *et al.*'s α (Moffatt 1978, p. 165). For vanishing Q, i.e. infinite conductivity, this α attains the finite value

$$\alpha = \frac{-1}{12\pi\bar{\rho}c^2 H} \int_0^\infty \int_{-\pi}^{\pi} \int_0^{\pi} \frac{E(k)\,[(\mathbf{k}.\hat{\mathbf{z}})\,(\mathbf{\Omega}.\mathbf{k}) + (\mathbf{\Omega}.\hat{\mathbf{z}})\,(1-\omega^2)]}{2\omega^4 - (1+k^2)\,\omega^2 - (\mathbf{k}.\mathbf{\Omega})^2 - (\mathbf{\Omega}.\hat{\mathbf{z}})^2} \sin\theta\, d\theta\, d\phi\, dk. \tag{54}$$

The fact that a finite α-tensor exists despite the vanishing helicity can be explained in the following way. The two waves of opposite helicity, which propagate into the (θ, ϕ) and $(\pi - \theta, -\phi)$ directions, induce fields such that their respective contributions to $\mathscr{E}$ do not cancel. This is essentially due to the effect of stratification; the two induced field waves have the same frequency, but, as can be seen from (41) with (39) and (42), they differ in their phase relationship relative to their inducing $\mathbf{u}$-waves.

We have so far assumed a constant mean magnetic field $\bar{\mathbf{h}}$. A spatially variable $\bar{\mathbf{h}}$ could however substantially modify our result. In the following particular case, where the Alfvén velocity is constant, the α-effect is even zero. With $\bar{\mathbf{h}} = \bar{\mathbf{h}}_0 \exp(z/2H)$ we obtain instead of (40) and (41) the induced field

$$\mathbf{h} = \mathrm{Re}\,[\hat{\mathbf{h}} \exp i(\mathbf{k}'.\mathbf{x} - \omega' t)],$$

$$\hat{\mathbf{h}} = -\frac{\tilde{\mathbf{u}}(\mathbf{k}'.\bar{\mathbf{h}}_0) - \bar{\mathbf{h}}_0(\mathbf{k}'.\tilde{\mathbf{u}})}{\omega' + i\lambda k'^2}.$$

The asymmetric part in the denominator of the α-tensor then vanishes. The same operations which led to expressions α_1 and α above now give

$$\alpha_1 = \frac{Q}{8\pi\bar{\rho}c^2 H}\int_0^\infty\int_{-\pi}^{\pi}\int_0^\pi \frac{E(k)\,k^4\omega[\Gamma\omega\sin\phi\cos\phi - 2(\mathbf{\Omega}.\mathbf{k})\cos^2\phi]\sin^3\theta\,d\theta\,d\phi\,dk}{[2\omega^4 - (1+k^2)\,\omega^2 - (\mathbf{k}.\mathbf{\Omega})^2 - (\mathbf{\Omega}.\hat{\mathbf{z}})^2]\,(\omega^2 + Q^2k^4)}$$

and

$$\alpha = \frac{Q}{12\pi\bar{\rho}c^2 H}\int_0^\infty\int_{-\pi}^{\pi}\int_0^\pi \frac{E(k)\,k^2\omega[(\mathbf{\Omega}.\mathbf{k})\,(\omega^2 - k^2) - (\mathbf{\Omega}.\hat{\mathbf{z}})\,(\mathbf{k}.\hat{\mathbf{z}})]\sin\theta\,d\theta\,d\phi\,dk}{[2\omega^4 - (1+k^2)\,\omega^2 - (\mathbf{k}.\mathbf{\Omega})^2 - (\mathbf{\Omega}.\hat{\mathbf{z}})^2]\,(\omega^2 + Q^2k^4)}.$$

Both expressions are zero because of the anti-symmetry of the integrands; cf. the argument given in §4 above concerning the mean helicity (we thank a referee for drawing our attention to this case). A non-vanishing α-tensor is however obtained whenever the mean field is not proportional to $\exp(z/2H)$. In particular, in cases where $\bar{\mathbf{h}}$ varies only on scales much larger than H our result (53) remains approximately valid. Some properties and consequences of this result are discussed in the following sections.

7. Approximative results

In this section we shall present some results obtained in the limit of slow rotation, $\Omega \ll 1$. Moreover, we shall only consider the leading terms of the expansions of the α-tensor in terms of Q and Q^{-1}, respectively. The α-tensor will be given for both acoustic and internal gravity waves, but the energy density spectrum will be assumed to have the form of a δ function, i.e.

$$E(k') = \frac{\bar{\rho}cu^2}{2k_p'}\,\delta\left(1 - \frac{k'}{k_p'}\right), \tag{55}$$

where the normalization is such that u is the r.m.s. velocity (after integrating over k', we drop the subscript p). Since the resulting expressions are still rather complicated, we restrict ourselves further to the cases of very small and very large wavenumbers k. Only the acoustic waves in the limit of large conductivity, $Q \ll 1$, will be considered for all k.

In order to obtain the various approximations, Wälder (1978) has calculated the appropriate Taylor expansions of the frequencies and of the integrand of (48). We shall not repeat these lengthy calculations here, but summarize only the results. First the frequencies: the acoustic branches are, for $k \gg 1$,

$$\omega_{K\pm} = \pm\left(k + \frac{1}{2k} - \frac{N^2(\mathbf{k}\times\hat{\mathbf{z}})^2}{2k^3}\right) - \frac{\Gamma\mathbf{k}.(\hat{\mathbf{z}}\times\mathbf{\Omega})}{2k^2}, \tag{56}$$

and, for $k \ll 1$,

$$\omega_{K\pm} = \pm[1 + \tfrac{1}{2}k^2 - \tfrac{1}{2}(\mathbf{\Omega}.\hat{\mathbf{z}})^2 - \tfrac{1}{2}N^2(\mathbf{k}\times\hat{\mathbf{z}})^2] - \tfrac{1}{2}\Gamma\mathbf{k}.(\hat{\mathbf{z}}\times\mathbf{\Omega}). \tag{57}$$

	Acoustic waves $k \gg 1$	Acoustic waves $k \ll 1$	Internal gravity waves $k \gg 1$
α_1	$\frac{8}{15}(\mathbf{\Omega}.\hat{\mathbf{z}})\,Q^2$	$-\frac{8}{15}(\mathbf{\Omega}.\hat{\mathbf{z}})\,Q^2k^4$	$-\frac{4}{3}(\mathbf{\Omega}.\hat{\mathbf{z}})\,Q^2k^4\delta^3$
α_2	$(\frac{5}{6}\Gamma^2-2)(\mathbf{\Omega}.\hat{\mathbf{z}})\,k^{-4}$	$-2(\mathbf{\Omega}.\hat{\mathbf{z}})$	$-(\mathbf{\Omega}.\hat{\mathbf{z}})\,N^2\delta$
α_4	$\frac{2}{3}k^{-2}$	1	$\frac{1}{2}N^2\delta$
α_5	$\frac{4}{15}\Gamma(\mathbf{\Omega}.\hat{\mathbf{z}})\,Qk^{-2}$	$\frac{7}{6}\Gamma(\mathbf{\Omega}.\hat{\mathbf{z}})\,Qk^2$	$\frac{1}{8}\Gamma(\mathbf{\Omega}.\hat{\mathbf{z}})\,Qk^2\delta$
α_6	$\frac{11}{30}\Gamma Qk^{-2}$	$-\frac{5}{6}\Gamma Qk^2$	$-\frac{1}{8}\Gamma Qk^2\delta$
α_7	$\frac{7}{13}Q$	$-Q$	$Qk^2N^2\delta$
α_8	$-\frac{1}{15}(\mathbf{\Omega}.\hat{\mathbf{z}})\,Qk^{-2}$	$(\mathbf{\Omega}.\hat{\mathbf{z}})\,Q$	$(\mathbf{\Omega}.\hat{\mathbf{z}})\,Qk\delta(\frac{3}{8}\Gamma^2+N^2+\frac{2}{3}k^2\delta^2)$
α_9	$-\frac{1}{3}\Gamma k^{-2}$	Γk^2	$-\frac{1}{4}\Gamma N^2\delta$

TABLE 1. Coefficients of the α-tensor, according to (52), for large conductivity, $Q \ll 1$; a factor $u^2/2c$ is to be added to all expressions

	Acoustic waves $k \gg 1$	Acoustic waves $k \ll 1$	Internal gravity waves $k \gg 1$	Internal gravity waves $k \ll 1$
α_1	$\frac{8}{15}\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q^2k^4}$	$-\frac{8}{15}\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})\,k^4}{Q^2}$	$-\frac{8}{15}\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q^2k^4}$	$\frac{8}{15}\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})\,k^4}{Q^2}$
α_2	$-\frac{2}{5}\frac{\Gamma^2(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q^2k^2}$	$-\frac{2(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q^2}$	$-\left(\frac{14}{5}+\frac{\Gamma^2}{30}\right)\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q^2k^6}$	$\frac{2(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q^2}$
α_4	$\frac{\Gamma^2}{5Q^2k^2}$	$\frac{1}{Q^2}$	$-\frac{1}{5Q^2k^4}$	$\frac{\Gamma^2+4-8N^2}{12Q^2}k^2$
α_5	$-\frac{37}{30}\frac{\Gamma(\mathbf{\Omega}.\hat{\mathbf{z}})}{Qk^6}$	$\frac{5\Gamma(\mathbf{\Omega}.\hat{\mathbf{z}})\,k^2}{Q}$	$\frac{\Gamma(\mathbf{\Omega}\cdot\hat{\mathbf{z}})}{6N^2Qk^2}$	$\frac{\Gamma(\mathbf{\Omega}.\hat{\mathbf{z}})}{4N^2Q}$
α_6	$\frac{11}{30}\frac{\Gamma}{Qk^4}$	$-\frac{\Gamma k^2}{2Q}$	$-\frac{\Gamma}{6N^2Qk^2}$	$\frac{\Gamma k^2}{6Q}$
α_7	$-\frac{1}{3Qk^2}$	$-\frac{1}{Q}$	$\frac{2}{3Qk^2}$	$\frac{2(N^2-1)\,k^2}{3Q}$
α_8	$(\Gamma^2-\frac{7}{3})\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{5Qk^4}$	$\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q}$	$\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{3N^2Qk^2}$	$-\left(\frac{\Gamma^2}{8N^2}+1\right)\frac{(\mathbf{\Omega}.\hat{\mathbf{z}})}{Q}$
α_9	$\frac{2\Gamma}{5Q^2k^2}$	$\frac{\Gamma k^2}{3Q^2}$	$-\frac{\Gamma}{6Q^2k^4}$	$-\frac{\Gamma k^2}{3Q^2}$

TABLE 2. Coefficients of the α-tensor, according to (52), for small conductivity, $Q \gg 1$; a factor $u^2/2c$ must be added throughout

The internal gravity wave branches are, for $k \gg 1$,

$$\omega_{G\pm} = \pm\frac{N}{k}|\mathbf{k}\times\hat{\mathbf{z}}| + \frac{\Gamma\mathbf{k}.(\hat{\mathbf{z}}\times\mathbf{\Omega})}{2k^2} \tag{58}$$

and, for $k \ll 1$,

$$\omega_{G\pm} = \pm[(\mathbf{\Omega}.\hat{\mathbf{z}})^2+N^2(\mathbf{k}\times\hat{\mathbf{z}})^2]^{\frac{1}{2}} + \tfrac{1}{2}\Gamma\mathbf{k}.(\hat{\mathbf{z}}\times\mathbf{\Omega}). \tag{59}$$

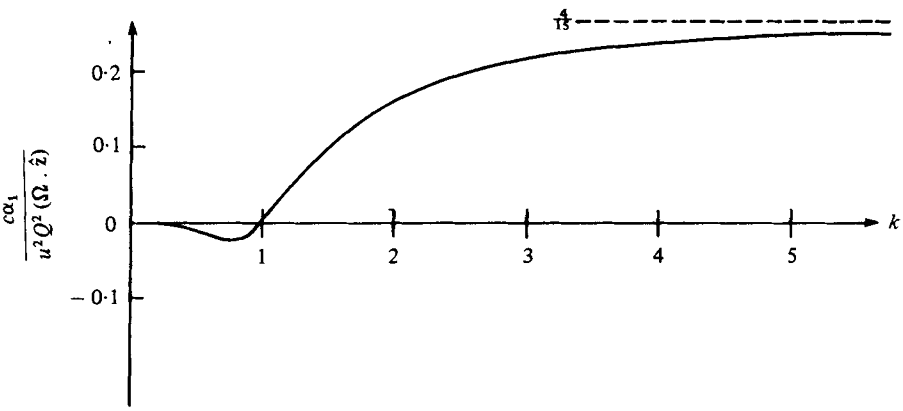

FIGURE 4. The α_1-effect of acoustic waves as a function of wavenumber, for $\Omega \ll 1$ and $Q \ll 1$, according to (60).

The results for the α-tensor are given in form of the coefficients $\alpha_1, \ldots, \alpha_9$ defined by (52), in table 1 for the limit of large conductivity, $Q \ll 1$, and in table 2 for the small conductivity case, $Q \gg 1$.

The α_3 term turns out to be of third order in the rotation rate and of second order in Q or Q^{-1}, respectively, and is therefore not included in tables 1 and 2. Also, in table 1 we did not consider internal gravity waves of small wavenumber for the following reason: if both Ω and k are small, the frequency ω of these waves, by (59), will approach zero and if, in addition, Q is small, the denominator of the fluctuating magnetic field amplitude (41) is also small, and the approximation of small fluctuations breaks down. For the same reason we restricted the large wavenumber internal gravity waves in table 1 to waves propagating almost horizontally, i.e. to small angles $\delta = \frac{1}{2}\pi - \theta$. The frequency of these waves is then close to the Brunt–Väisälä frequency N.

For acoustic waves and $Q \ll 1$, the coefficient α_1 has been computed by Wälder (1978) for all wavenumbers. She obtained

$$\alpha_1 = \frac{Q^2(\mathbf{\Omega}.\hat{\mathbf{z}})\,u^2}{4N^3c}\,k(k^2-1)\left(\frac{6}{a}+\frac{3+a^2}{a^2}\ln\frac{1-a}{1+a}-\frac{4}{a}\ln(1-a^2)\right), \qquad (60)$$

where $a = 2Nk/(1+k^2)$. The approximative formulas of table 1 can be recovered from (60) by an expansion in terms of a, including orders up to a^5. Figure 4 shows α_1 as a function of wavenumber. The change of the sign of α_1 at $k = 1$ occurs also for some of the other components of the α-tensor, as can be seen from tables 1 and 2 in the various cases.

8. A two-dimensional mean field: conclusions

In this section we shall investigate some of the consequences of the α-tensor derived in the preceding sections on the mean field. We are mainly interested in the differences from the case of a scalar α-effect, and shall therefore neglect the dependence of the α-tensor on $\cos\theta_0$ and $\bar{p}$. This is not consistent in principle; moreover, it means that we neglect a variation of the mean field which we should have taken into account in

§§ 5 and 6 where we derived the α-tensor in the first place. The following is nevertheless added here in order to demonstrate differences which obtain from the tensorial form of α, and to make order of magnitude estimates concerning possible stellar dynamos. In the case $k \gg 1$, where the wavelength is short compared to the scale height of stratification, such estimates might indeed be sufficiently accurate. The mean field induction equation is, then,

$$\frac{\partial \bar{h}_i}{\partial t} = \epsilon_{ijk} \alpha_{kl} \frac{\partial \bar{h}_l}{\partial x_j} + \lambda \Delta \bar{h}_i \tag{61}$$

(e.g. Moffatt 1978, p. 202). In analogy to the axi-symmetric mean field in the spherical case, we seek a field which is independent of the direction $\boldsymbol{\Omega} \times \hat{\mathbf{z}}$, i.e. of the co-ordinate x. This field has the form

$$\bar{\mathbf{h}} = \operatorname{curl}(A\hat{\mathbf{x}}) + B\hat{\mathbf{x}}, \tag{62}$$

where $\hat{\mathbf{x}}$ is a unit vector, and A and B have exponential form

$$(A, B) = (A_0, B_0)\, e^{i\mathbf{m}.\mathbf{x}+nt}. \tag{63}$$

n is the growth rate (generally complex), and $\mathbf{m}$ the wave vector, which is perpendicular to $\hat{\mathbf{x}}$. We insert (62) and (63) in (61) and obtain

$$[n + \lambda m^2 + iR - i(\mathbf{v}_p.\mathbf{m})]\, B_0 = \frac{S}{\alpha_1} A_0, \tag{64}$$

$$[n + \lambda m^2 - iR - i(\mathbf{v}_p.\mathbf{m})]\, A_0 = \alpha_1 B_0. \tag{65}$$

The vector $\mathbf{v}_p = \alpha_7 \hat{\mathbf{z}} + \alpha_8 \boldsymbol{\Omega}$ is equivalent to a poloidal mean flow; we have also introduced

$$R = \alpha_5(\mathbf{m} \times \hat{\mathbf{x}}).\hat{\mathbf{z}}(\boldsymbol{\Omega} \times \hat{\mathbf{z}}).\hat{\mathbf{x}} + \alpha_6(\mathbf{m} \times \hat{\mathbf{x}}).\boldsymbol{\Omega}(\boldsymbol{\Omega} \times \hat{\mathbf{z}}).\hat{\mathbf{x}}, \tag{66}$$

$$S = \alpha_1^2 m^2 + \alpha_1 \alpha_2 [(\mathbf{m} \times \hat{\mathbf{x}}).\hat{\mathbf{z}}]^2 + 2\alpha_1 \alpha_4 (\mathbf{m} \times \hat{\mathbf{x}}).\hat{\mathbf{z}}(\mathbf{m} \times \hat{\mathbf{x}}).\boldsymbol{\Omega}. \tag{67}$$

The term $\mathbf{v}_t = \alpha_9 \boldsymbol{\Omega} \times \hat{\mathbf{z}}$, which is equivalent to a toroidal mean flow, does not occur in (64) and (65) since it is constant everywhere and thus does not affect a field which is independent of the direction $\mathbf{v}_t$. The condition for the existence of non-trivial solutions yields the growth rate

$$n = -\lambda m^2 + i(\mathbf{v}_p.\mathbf{m}) \pm (S - R^2)^{\frac{1}{2}}. \tag{68}$$

For the upper sign we have dynamo action provided

$$S - R^2 > \lambda^2 m^4. \tag{69}$$

From (66) and (67) and tables 1 and 2 we see that $R^2 \propto \Omega^4$ and $S \propto \Omega^2$. Dynamo action appears therefore possible in the case of small Ω, because in an infinitely extended fluid (69) can always be satisfied if only the left-hand side is positive. In a finite geometry, however, m will be determined by the inverse of the length scale, and a more detailed consideration of the involved parameters, mainly Ω, Q, and k, is necessary. In any case it is clear from (64) and (65) that α_1 is the crucial link for a regenerative dynamo of the type described here.

The mean field (63) has the form of a propagating wave, with frequency $-(\mathbf{v}_p.\mathbf{m})$. Although the phase of such a wave propagates in all directions except perpendicular to $\mathbf{v}_p$, its group velocity is simply $-\mathbf{v}_p$, which in the limit of small rotation is approximately the vertical direction, upward or downward depending on the sign of α_7.

The ratio of the field components in the x direction and in the direction perpendicular to $\hat{\mathbf{x}}$ is $|B_0|/m|A_0|$. Using (68) we find from (64) and (65)

$$\frac{|B_0|^2}{m^2|A_0|^2} = \frac{S}{m^2\alpha_1^2} = 1 + \frac{\alpha_2[(\mathbf{m}\times\hat{\mathbf{x}}).\hat{\mathbf{z}}]^2}{\alpha_1 m^2} + \frac{2\alpha_4}{\alpha_1}\frac{[(\mathbf{m}\times\hat{\mathbf{x}}).\hat{\mathbf{z}}][(\mathbf{m}\times\hat{\mathbf{x}}).\mathbf{\Omega}]}{m^2}.$$

For large conductivity, i.e. $Q \ll 1$, the results of table 1 indicate that α_2/α_1 and α_4/α_1 are both proportional to $Q^{-\frac{2}{1}}$, so that the mean field is essentially in the x direction. On the other hand, for $Q \gg 1$, $S/m^2\alpha_1^2$ depends on k, but not on Q and Ω, in the approximative results of table 2; the two components of the mean field are then comparable in magnitude for internal gravity waves at large k, while we still have $|B_0| \gg m|A_0|$ for the other cases.

Can the waves discussed in this paper play a role in driving stellar dynamos? For the Sun, we may exclude internal gravity waves, since these can only exist where the stratification is stable, i.e. either below the convection zone or in the atmosphere itself. Certainly the latter is not the seat of the solar dynamo since observation shows that magnetic flux continuously emerges from the interior. On the other hand, a dynamo below, say, 2×10^8 m, the depth of the convection zone in current solar models (e.g. Baker & Temesváry 1966), poses problems because both the time scales of oscillation (e.g. Stix 1976) and of the emergence of magnetic flux to the surface (Parker 1975) would be much too long.

Acoustic waves with periods around 5 min have been observed for many years in the solar atmosphere, and more recently it became clear that the origin of these waves must lie in the upper part of the convection zone (Ulrich 1970; Ando & Osaki 1975; Deubner 1975). There typical values of H and c are 500 km and 10 km/s, respectively (Baker & Temesváry 1966), so that, by (11), we have $\Omega = 6\times10^{-4}$. Q is also small; we use a diffusivity $\lambda_t \approx u_c l/3$, where u_c and l are velocity and scale of the convection. (The use of a turbulent electromagnetic diffusivity in the present context really means that we have a 'three-scale-picture' in mind. On the smallest scale, we imagine isotropic, mirror-symmetric turbulence having the sole effect of a modified diffusion coefficient. Only at intermediate scales do the anisotropies introduced by rotation and stratification become important, leading to our α-tensor). With $l = H$, $u_c = 10^3$ m/s (Baker & Temesváry 1966), and the values already used we find, from (45), $Q = 1{\cdot}5\times10^{-2}$. Typical vertical wave lengths of the solar oscillations are a few thousand kilometres (Knölker 1978). Since there is a whole spectrum of waves, we cannot immediately use figure 4 in order to obtain α_1; in particular, the sign reversal at $k = 1$ creates uncertainty. However, for an order of magnitude estimate it is sufficient to take $\alpha_1 \approx 0{\cdot}1\,Q^2\Omega u^2/c$. The r.m.s. wave amplitude observed at the surface is a few hundred metres per second; the amplitude decreases inwards so that $u = 100$ m/s appears to be an optimistic estimate. With this we have $\alpha_1 \approx 1{\cdot}4\times10^{-8}$ m/s. For α_2 and α_4 we are content with an even simpler estimate: we multiply α_1 by Q^{-2} and obtain $\alpha_2 \approx \Omega\alpha_4 \approx 6\times10^{-3}$ m/s. In order to see whether this is sufficient for dynamo action, we consider a mean field wave having a wavelength comparable to the solar radius. Then $m \approx 10^{-8}$ m^{-1} and $S^{\frac{1}{2}} \approx m(\alpha_1\alpha_2)^{\frac{1}{2}} \approx 10^{-14}$ s^{-1}. On the other hand $\lambda m^2 \approx 10^{-8}$ s^{-1}. That is, dynamo action clearly does *not* occur. Even in co-operation with non-uniform rotation α_1 appears to be far too small: For an '$\alpha_1\omega$-dynamo', α_2 would be replaced by $R\Delta\Omega'$, which on the Sun is approximately 100 m/s, still insufficient for dynamo instability.

Of course the convective motions on the Sun themselves are strongly distorted by the effects of rotation, although the α-effect derived from these motions (Krause 1968; Yoshimura 1972) still rests on the first-order smoothing approximation in a case where it is not well justified. But the values of α obtained by this procedure are of order 1 ms^{-1} (e.g. Stix 1976), so it appears that the Sun, after all, does not need a wave-driven dynamo. Whether or not other stars can have a dynamo of the type discussed here largely depends on the presence of a turbulent convection zone. If there is none, the molecular conductivity has to be used. Q would then be very large and, accordingly, the α-tensor much too small (e.g. Deinzer 1976). If turbulence is present, and the rotation rate and wave amplitude are both larger, say, by a factor of 10 than the above-adopted values, a stellar $\alpha_1\omega$-dynamo might exist. Stellar oscillations with such an amplitude could be detectable with interferometric techniques (Traub, Mariska & Carleton 1978). Another possible field of application is waves in planetary interiors.

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REFERENCES

ANDO, H. & OSAKI, Y. 1975 *Publs astron. Soc. Japan* **27**, 581.

BAKER, N. & TEMESVÁRY, S. 1966 *Tables of Convective Stellar Envelope Models*. 2nd edn. New York: N.A.S.A.

DEINZER, W. 1976 *Mitt. astr. Ges., Hamb.* **40**, 156.

DEUBNER, F. L. 1975 *Astron. & Astrophys.* **44**, 371.

ECKART, C. 1960 *Hydrodynamics of Oceans and Atmospheres*. Pergamon.

KNÖLKER, M. 1978 *Oszillationen der ganzen Sonne–Modellrechnungen*. Diplomarbeit, University of Göttingen.

KRAUSE, F. 1968 *Eine Lösung des Dynamoproblems auf der Grundlage einer linearen Theorie der magnetohydrodynamischen Turbulenz*. Habilitationsschrift, University of Jena.

MOFFATT, H. K. 1970*a* *J. Fluid Mech.* **41**, 435.

MOFFATT, H. K. 1970*b* *J. Fluid Mech.* **44**, 705.

MOFFATT, H. K. 1972 *J. Fluid Mech.* **53**, 385.

MOFFATT, H. K. 1974 *J. Fluid Mech.* **65**, 1.

MOFFATT, H. K. 1978 *Magnetic field generation in electrically conducting fluids*. Cambridge University Press.

PARKER, E. N. 1955 *Astrophys. J.* **122**, 293.

PARKER, E. N. 1975 *Astrophys. J.* **198**, 205.

STEENBECK, M., KRAUSE, F. & RÄDLER, K.-H. 1966 *Z. Natur.* **21a**, 369.

STIX, M. 1976 *Dynamo theory and the solar cycle*. In *Basic Mechanisms of Solar Activity*, IAU Symp. 71 (ed. V. Bumba & J. Kleczek), p. 367.

TRAUB, W. A., MARISKA, J. T. & CARLETON, N. P. 1978 *Astrophys. J.* **223**, 583.

ULRICH, R. K. 1970 *Astrophys. J.* **162**, 993.

WÄLDER, M. 1978 *Dynamowirkung von Schall- und Schwerewellen in geschichteten rotierenden Atmosphären*. Dissertation, University of Göttingen.

YOSHIMURA, H. 1972 *Astrophys. J.* **178**, 863.